Exploring Chaos In A Damped, Driven Pendulum

Many non-linear systems are mathematical models for physics that is very difficult to handle. It may be a very unstable system, or a very complicated system. The mathematical model may be hard, or more likely impossible to solve exactly. The study of these systems brings a new kind of mathematics into the spotlight. This study is known as chaos theory.

One of the trademarks of a chaotic system is sensitive dependence on initial conditions. This alone does not mean that a system is chaotic, but every chaotic system has this property. Take for example a simple pendulum. For small oscillations the motion of the pendulum can be determined very easily as a sine function. If this pendulum were started from a set of initial conditions many times, the behavior would be roughly the same for each of the trials. Chaotic systems do not have this property. The smallest change in initial conditions could produce wildly different behavior. They seem to move erraticly, and any small perturbation could change the whole system drastically. However, these systems do have an underlying order. Finding and studying that order is a big part of the study of chaos.

-- Blair Fraser

Physics A-Level Practical Investigation

Exploring Chaos In A Damped, Driven Pendulum

Aim

The pendulum is a fairly simple non-linear system, although it is not so simple to solve. Its behavior is very familiar to everyone. The aim of this investigation is to demonstrate various situations in which a damped, driven pendulum may show signs of chaotic properties - something that is not so familiar to everyone. The results obtained from the apparatus will be compared to other forms of chaotic behavior. Ways of identifying chaotic behavior will also be explained and demonstrated.

Summary

In this investigation a mechanical implementation of a damped, driven pendulum is set up and some of its properties are explored for signs of chaos. Information about the pendulum is recorded in the form of the angle of deflection, the frequency of the driving force and the magnitude of the driving force that is applied to the pendulum. The frequency of the driving force remains constant during the investigation and the effect that varying the magnitude of the driving force has on the angle of deflection is measured.

There are several different mathematical expressions available for modeling a pendulum. The properties and uses of two models are shown below;

The linear equation for the motion of a pendulum is;

$$a\sin(\omega t + \varepsilon) = x$$

where a is the amplitude, ω is the angular frequency, ε is the starting point and x is the displacement at time t.

However, this equation is only suitable for oscillations that involve small angles of displacement. This is because the period is always the same, no matter how large the angle of deflection is. This is acceptable as an approximation for small angles of deflection, where $\sin\theta \approx \theta$, but when oscillations occur where θ is larger than φ, this model breaks down.

In this investigation the pendulum will be going all the way round. The pendulum will also be damped and driven. A second order differential equation will be used to model the pendulum. A differential equation can model non-linear systems. The equation that will be used is;

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta - q\frac{d\theta}{dt} + F_D\sin(\omega_D t)$$

where g is acceleration due to gravity, l is the length of the pendulum, F_D is the magnitude of the driving force, ω is the frequency of the driving force, t is time, θ is the angular displacement of the pendulum in radians, q is the damping parameter for the pendulum, $\frac{d^2\theta}{dt^2}$ is the (angular) acceleration of the pendulum and $\frac{d\theta}{dt}$ is the (angular) velocity of the pendulum.

The pendulum system has two fixed points, one is stable and the other is unstable. The stable point is at position = 0.0 and velocity = 0.0. This corresponds to a pendulum hanging straight down and at rest. The other fixed point is unstable and is at position = 3.14159^c and velocity = 0.0. This unstable fixed point is a pendulum resting straight up. The fact that the pendulum is not moving is very important - when the pendulum is moving it may pass through these points, but it is not stable.

The pendulum has two '*degrees of freedom*'. These *degrees of freedom* are its velocity and its position. - These are all the things that there are to know about the pendulum at a certain instant.

The damping parameter for the pendulum in this experiment is unknown because it was not possible to experimentally determine this value with the equipment available. The frequency of the driving force was set at 0.5Hz. The way in which this value was determined is explained in the '*Diary of events*'

Diary of events

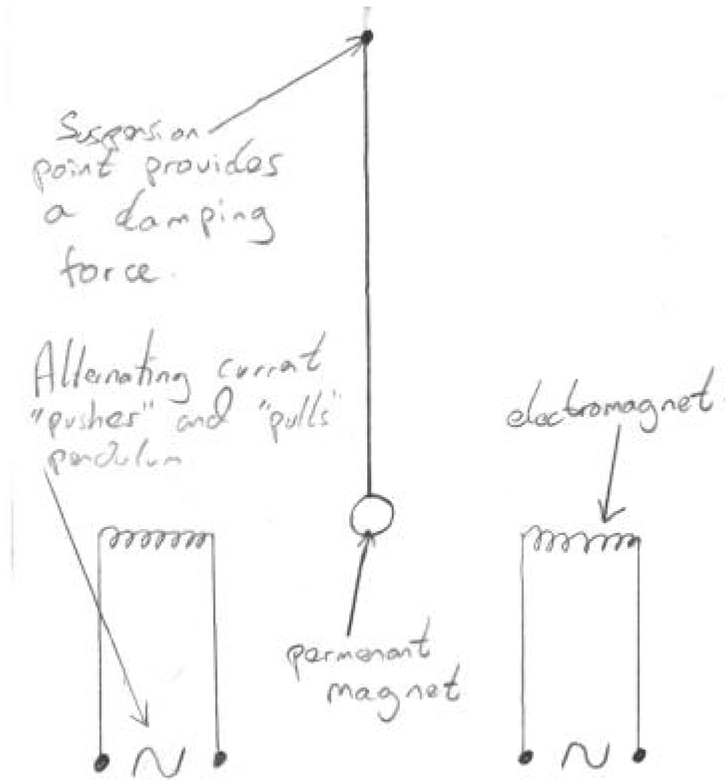
Plan

Two weeks were available for lab work. The time was split up in the following way;
Week 1 - Construction, set up and testing of the apparatus.
Week 2 - Obtain readings and measurements from the apparatus.

NOTE: Events and observations that occurred during lab time are written in the past tense. Any additional comments that were made at the time of writing the diary entry are written in the present tense.

Daily entries, comments and observations

Day 1 - 25-9-2000 - Monday



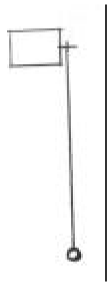
It was proposed that the pendulum could be driven by a pair of electromagnets - one on each side of the pendulum. However, the electromagnet that was constructed did not provide a large enough force. It was then proposed that the pendulum was driven with a motor. This method showed potential although the prototype motor was not strong or smooth enough. Although the smoothness of a motor can be improved by adding more commutators, this would involve constructing a motor from scratch.
Construction was started on a pendulum. It currently consists of a long wooden dowel that can be cut to the required length when the driving force is finalised. An executive toy was observed that consisted of a magnetic pendulum driven by an electromagnet, so maybe our design for the configuration of the electromagnet needs to be changed.

Day 2 - 26-9-2000

The idea of driving the pendulum with an electromagnet was developed further. Attempts were made to simulate the configuration of an "executive toy" but the electromagnet was still unable to provide a large enough force to drive the pendulum in the required way.
A suitable motor was found that was able to drive the pendulum, although there was not enough time to conduct any tests.

Day 3 - 27-9-2000

Extensive tests were conducted on the motor today. It was able to drive the pendulum in circles. However, the dowel was did not behave well as a pendulum. A mass was added to the bottom of the pendulum to combat this. The mass needed to be large enough to overcome the large damping effect of the motor, but the motor was not strong enough to drive the new pendulum round. A larger current was passed through the motor but this did not solve the problem.



It was decided that a stronger motor was needed. To do this the motor would have to have stronger permanent magnets. - This however, would increase the damping effect of the motor which was not wanted. A motor was needed that possessed electromagnets. These had to be wired separately from the driving coil so that the motor could be driven both clockwise and anticlockwise.

Day 4 - 28-9-2000

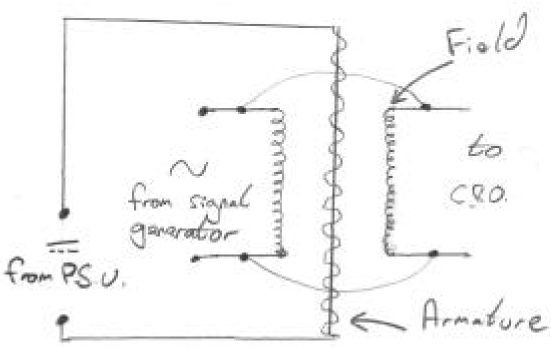
A suitable motor was found today, although it did not meet the exact specifications that were outlined yesterday. The motor had three terminals - Common, Forward and Reverse. So although the motor could be driven in both directions, the electromagnets were connected directly to the driving coil. It was proposed to drive the motor with a change-over relay. The signal generator would switch the relay, and a power pack would be connected through the relay to the motor. This proved successful. However, the signal that was fed to the motor was in the form of a square wave. This meant that the movement of the pendulum was not very smooth.

Day 5 - 29-9-2000

An attempt was made to 'weight' the pendulum. Various lengths of pendulum were tested.

Day 6 - 30-9-2000

The motor was changed for the third time because a motor was found with the exact specification as set out earlier. - Separate connections for the armature and the field were available. A sine wave from the signal generator was fed into the field of the motor and an oscilloscope. The armature of the motor was fed from a variable voltage power supply. This means that the force applied to the pendulum can be changed by changing the voltage across the power pack. The frequency of the driving force is determined by the output from the signal generator.



The apparatus is now ready for readings to be taken. It is estimated that the investigation is on schedule.
Readings are obtained by noting the position of the pendulum at times that are in phase with the driving force. Through tests, it was found that the pendulum moved well when it was driven at

0.5Hz. A strobe was set up, using the oscilloscope, so that the apparatus was illuminated at the 'bottom' of each sine wave.

Day 7 - 2-10-2000

Attempts were made to take readings from the pendulum.

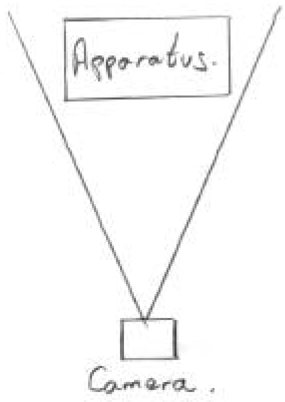
Day 8 - 3-10-2000

More attempts were made to obtain some results from the pendulum. However, it was difficult to record the position of the pendulum and even more difficult to take measurements from these recordings. It was found that it was not possible to get the strobe exactly in phase with the driving force. - The phase of the oscilloscope and the strobe would initially appear to be the same, but after about 15 seconds the phase difference became clear.

Day 9 - 4-10-2000

It was proposed to use a camera to record the position of the pendulum. A digital camera was obtained. - This meant that money and film would not be wasted on failed attempts. There was not enough time to begin tests because only one period was available.

Day 10 - 5-10-2000



The camera was set up and various automatic settings were tried, but it was easier to trigger the camera manually. It was necessary to move the pendulum into a larger room so that all of the pendulum would fit into the frame. It was also necessary to colour parts of the pendulum so that it would show up in the photo. Some encouraging results were obtained from the camera, but it was still difficult to distinguish the pendulum from the background, the camera could only take 10 pictures in quick succession - more than this are needed for big driving forces. It was also found that some of the pictures were out of phase. This is because it would take nearly one period (of the driving force) for the camera to take a picture after the button was pressed.

Day 11 - 6-10-2000

It was decided to use a video camera to record the pendulum. - This way the correct frames could be selected after the experiment. A video camera was obtained and the battery was left to charge.

Day 12 - 7-10-2000

The pendulum was filmed for various driving forces. The driving force was incremented in 0.5Volt steps. It is not known whether there is a linear relationship between voltage and force.

Day 13 - 9-10-2000

The filming of the pendulum was continued. The pendulum started to come loose from the motor. The tape had become quite hot during the experiments, so the apparatus was left to cool.

Day 14 - 10-10-2000

It was found that once the tape had cooled, the connection between the motor and the pendulum had improved. More readings were recorded from the pendulum. The pendulum was also filmed while the driving force was kept constant and the starting angle was varied. The pendulum began to work itself loose from the motor again, but plenty of measurements had already been recorded.

Final apparatus

The picture above is a typical frame from the camera that the results were measured from. The camera was lined up exactly in front of the motor, but parallax is largely irrelevant because it does not affect the angle of the pendulum.

NOTE: Positive angles are measured anticlockwise from the downward vertical and negative angles are measured clockwise from the downward vertical.

Results

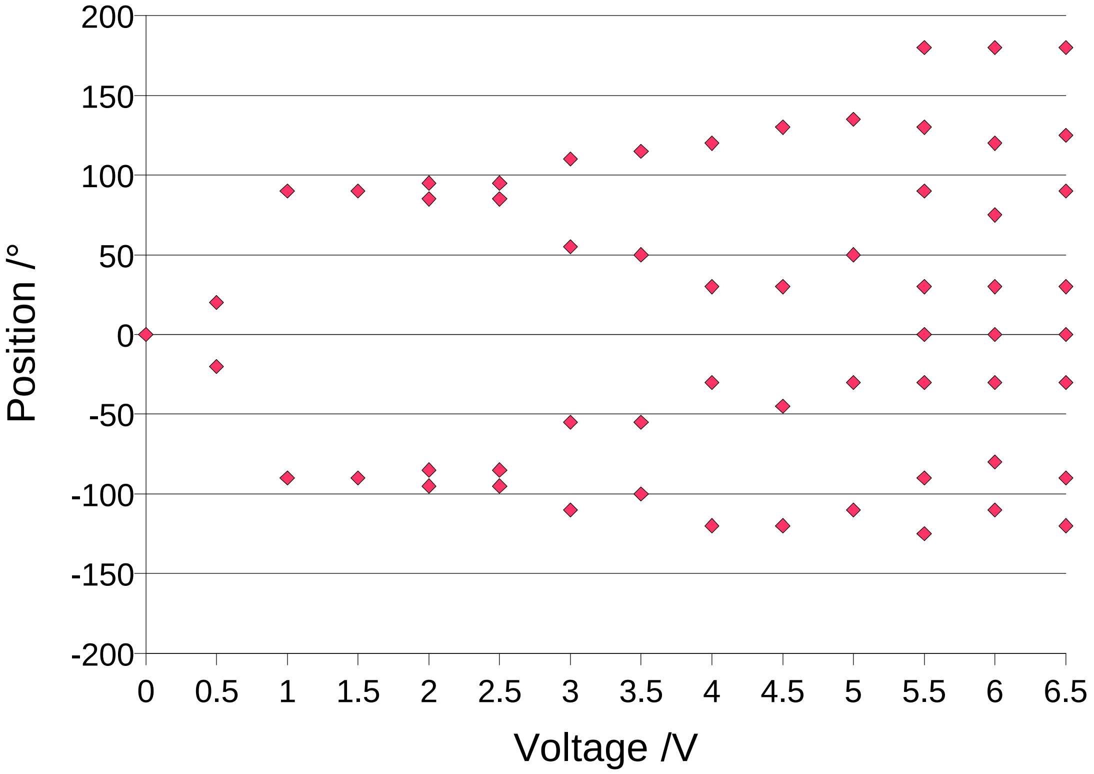
Experiment 1

For all of these experiments, the staring angle of the pendulum was 25°.

Voltage	Attempt	Position of pendulum at times in phase with the driving force /°							
0.5V	1	+20	-20						
	2	+20	-20						
1.0V	1	+30	-20						
	2	+90	-90						
1.5V	1	+85	-85						
	2	+90	-90						
2.0V	1	+85	-85	+95	-95				
	2	+85	-85	+95	-95				
2.5V	1	+85	-85	+95	-95				
	2	+85	-85	+110	-110				
3.0V	1	+55	-55	+110	-110				
	2	+55	-55	+110	-110				
3.5V	1	+50	-55	+115	-100				
	2	+65	-65	+155	-140				
4.0V	1	+30	-30	+115	-105				
	2	+30	-30	+120	-120				
4.5V	1	+65	-60	+85	-110				
	2	+30	-45	+130	-120				
5.0V	1	+80	-60	+125	-105				
	2	+80	-30	+110	-105				
	3	+50	-30	+135	-110				
5.5V	1	0	180	+30	-30	+90	-90	+130	-125
	2	0	180	+30	-30	+90	-90	+130	-125
6.0V	1	0	180	+30	-30	+75	-75	+130	-130
	2	0	180	+30	-30	+75	-80	+120	-110
	3	0	180	+30	-30	+75	-80	+120	-110
6.5V	1	0	180	+30	-30	+90	-90	+125	-120
	2	0	180	+30	-30	+90	-90	+125	-120
	3	0	180	+30	-30	+90	-90	+125	-120

The results were measured from the screen of a monitor, displaying the recordings from the video camera, with a protractor. As the video was played, points were marked on the screen at the 'top' and 'bottom' of the driving force. The state of the driving force could be seen on the oscilloscope that was recorded with the pendulum. The video was then paused and the points were measured and recorded along with the voltage.

Results that have been highlighted in red on the table do not appear to agree with the general pattern of results. The pattern that the results have produced can be seen more clearly in the graph on the next page. Results marked in red in the table have been omitted from the graph.



Experiment 2

To test the system's sensitivity to initial conditions, the starting angle of the pendulum was varied when the driving force was set to maximum. (6.5V).

Starting Angle /°	Voltage	Position of pendulum at times in phase with the driving force /°							
25	6.5V	0	180	+30	-30	+90	-90	+120	-120
30	6.5V	-60	+135	-135					
60	6.5V	-10	-25	-35	15	55	110	115	
85	6.5V	Results not available due to apparatus failure. (See Diary)							

Analysis

Experiment 1

The accuracy to which the angles were measured was ±5°. However, the quality and resolution of the video will undoubtedly increase the uncertainty. Therefore, the scale on the y-axis of the previous graph is such that the true value for the position lies somewhere within the 'dot' for each point. The voltage is the value that was shown on a volt meter that was connected across the motor (and therefore, the power supply). The load that the motor placed on the power supply was such that the setting on the front of the supply was much higher. The driving force was oscillating, so therefore, the motor did not draw a constant current. This meant that the voltage strayed above and below the recorded value. The voltage was carefully set for each experiment so that the recorded value was the average voltage across the motor. The large fields that the electromagnets in the motor produced caused 'ripples' to appear in the driving force, especially near the peaks. These 'ripples' where always present and the oscilloscope showed them to be the same for each experiment. Therefore, they can be ignored. - There is no way of accounting for their effect because they were always present, and therefore would only appear as a constant in the equation.

The graph shows a series of 'period doublings' This is a 'hallmark' of chaotic systems. A large number of chaotic systems exhibit this *period doubling cascade* and it is associated with a universal constant known as the *Feigenbaum constant*. The exact position of the period doubling is uncertain, especially around 2 to 3 volts. This period doubling will be taken as 3 volts.

The Feigenbaum constant

Period doublings occur just before a system enters the 'chaotic realm'. Using the Feigenbaum constant, if the position of two period doublings are known, the position of other period doublings can be calculated. This is can be illustrated with an example from '*Chaos*' by James Gleick;

...A line of identical telephone poles converges towards the horizon in a perspective drawing. If the size of two poles are known, then the size of the rest can be calculated. - The poles converge geometrically. - The period doublings 'accelerate' at a constant rate. This 'rate of acceleration' is known as the Feigenbaum constant and it has a value of 4.6692016090...

The ratios of the period doublings for the results from the pendulum can be calculated and compared to this value;

The results contain 3 period doublings, so only one ratio can be calculated. At the resolution of the results, period doublings occur at 0.5V, 3V and 5.5V. The horizontal distances between these points can be calculated.

$$3 - 0.5 = 2.5$$

$$5.5 - 3 = 2.5$$

$$\frac{2.5}{2.5} = 1$$

This result is very interesting because it was not known how the voltage across the motor was related to the driving force. - All that was known was that there was a relationship (See diary). The ratios between these period doublings is 1. Therefore the voltage must be connected to the force with the Feigenbaum constant.

To find this relationship it is necessary to compare the values for voltage and the values for force. Since the values for force are not known arbitrary units will be used. This value of force will be proportional to a force in Newtons.

$F_N =$ force in Newtons. $F_N \propto F_A$ therefore, $F_N = kF_A + c$

$F_A =$ force in arbitrary units.

$K_1 =$ force in arbitrary units at which the first period doubling occurs.

$K_2 =$ force in arbitrary units at which the second period doubling occurs.

$K_3 =$ force in arbitrary units at which the third period doubling occurs.

First, make the ratio of the forces equal to 4.66920

$K_2 - K_1 = 4.66920$ therefore, $K_2 = 4.66920 + K_1$

$K_3 - K_2 = 1$ therefore, $K_3 = 1 + K_2$

$$\frac{4.66920}{1} = 4.66920$$

Next, solve for values of K in arbitrary units

Let $K_1 = 1$

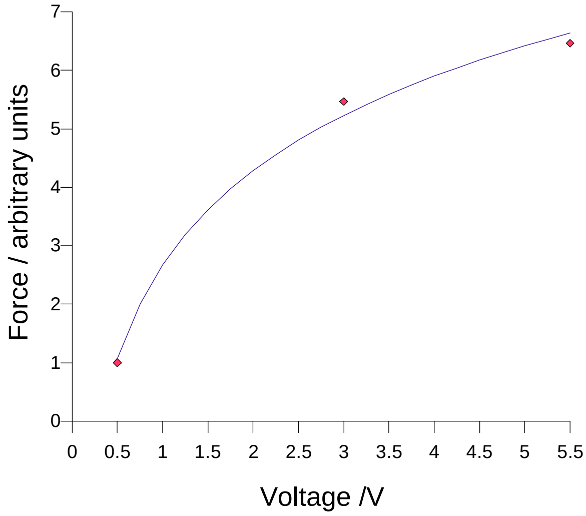
$1\ K_2 = 4.66920 + 1$ therefore, $K_2 = 5.466920$

$1\ K_3 = 1 + K_2$ therefore, $K_3 = 6.466920$

Now compare the calculated values for force with the voltage across the motor;

F_A	V
1	**0.5**
5.466920	**3.0**
6.466920	**5.5**

Below is a graph of the voltage across the motor against the force;

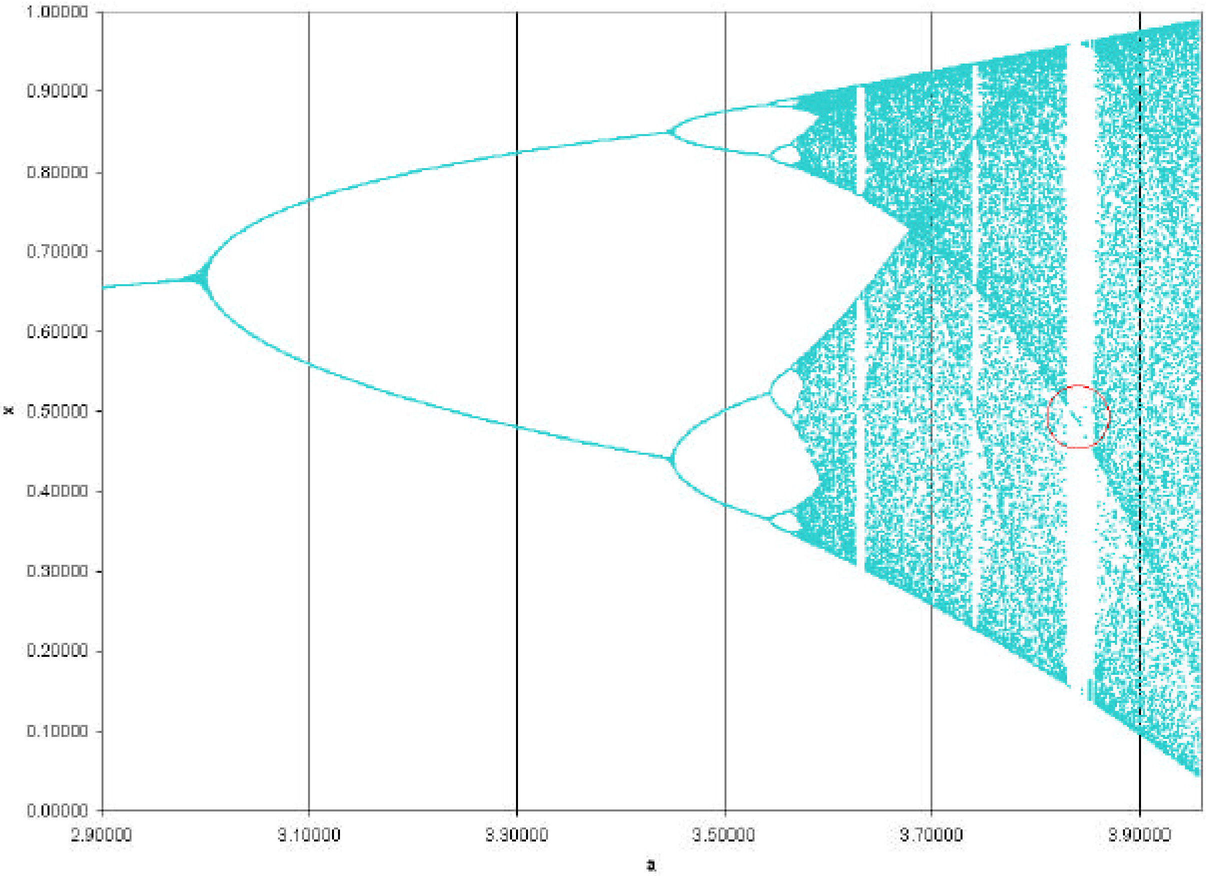


The line that has been drawn is a logarithmic regression that was calculated by the computer. It has the equation $y = 2.67 + 2.33 \ln(x)$. This is the best relationship that the computer could generate. However, it does not accurately map Voltage onto Force, so another equation needs to be found. - For an unloaded motor the relationship of Force to Voltage should be linear $1\ F = BIL$. In this experiment the current in the motor is not constant. - The current in the 'field' contains fluctuations and the current in the armature is sinusoidal.

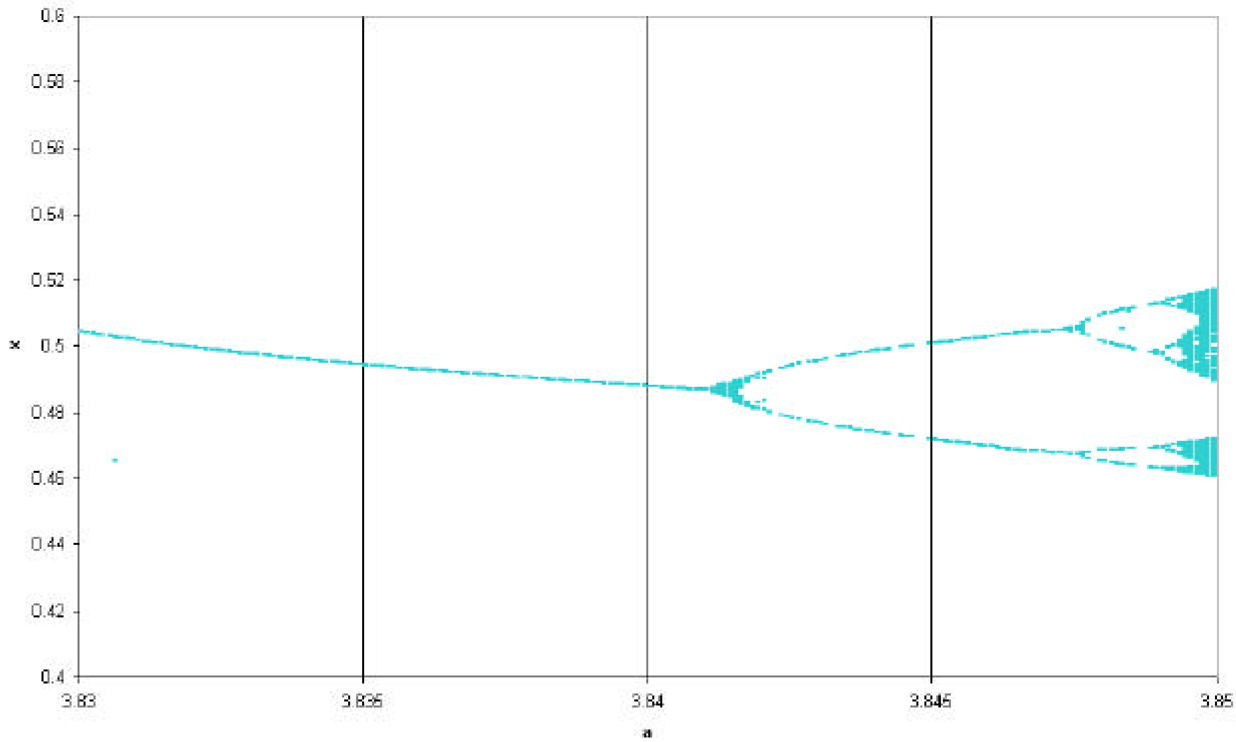
Period Doubling in other chaotic systems

The equation that is cited in this investigation as the equation of the pendulum is an example of a '*continuous map*'. This is caused by the differential coefficients. However, the same behavior can be seen in other, far simpler, '*discrete maps*'. One of the simplest chaotic maps is the *logistic map*. This map shows all of the characteristics that have been seen in the pendulum in this investigation.

The logistic map has the equation $x_{n+1} = ax_n(1 - x_n)$, where a takes values from 0 to 4. The value of a can be thought of as a similar quantity to the force applied to the pendulum. To generate the map, values for a are put into the equation. The map is then iterated by placing x_1 and a into the equation. For the purpose of this demonstration, x_1 is chosen to be 0.6, but other values can be chosen. The equation is iterated by taking the value that 'comes out' and placing it back into the equation. The first 50 iterations of the equation are discarded to allow the system to 'settle' just as in the pendulum experiments. The values of x that emerge after the system has 'settled' are plotted against the value of a on a graph. This process is then repeated for different values of a. The result of plotting x against a is shown below;



The period doublings that were observed in the pendulum experiment can be seen in the graph above. In the graph above, the distance between the period doublings are in the ratio 1:4.66920. The graph also appears to show some fractal properties. - The graph is self similar. When a = 3.83 the period doublings start again. This can be seen more easily if this part of the graph is shown separately;



The discontinuity of the lines is due to the resolution of the image

Although the experiments with the pendulum showed a '*period doubling cascade*', the power pack was unable to produce enough power to push the system into the chaotic realm. However, by conducting another experiment where the starting angle of the pendulum was varied it was possible to show the system's sensitivity to initial conditions.

Experiment 2

If the pendulum was not chaotic then the system would settle into a particular, stable, state and then remain there. If a 'jolt' was introduced to the system then the system would move into another, unstable, state and eventually return to the stable state. The system can be forced into a different state by varying the starting angle of the pendulum. We would expect the pendulum to eventually settle into the same state for each starting angle. From the results we can see that it does not do this. This is a chaotic characteristic. It should be noted that although everything that is chaotic shows this characteristic, not everything that shows this characteristic is chaotic. - i.e. Non-chaotic systems can show this characteristic. However, with the evidence from the previous experiment, it would not be unwise to attribute the observed behavior to chaos.

Conclusions

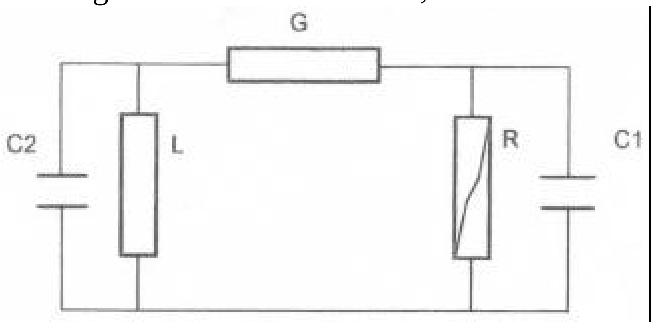
The investigation that was conducted was successful in demonstrating the presence of chaotic behavior in a damped, driven pendulum. However, it would have been useful to have collected more data for both experiments. In the first experiment data could be collected about the pendulum at a higher resolution, i.e. In 0.2V or 0.1V steps. The pendulum should also be investigated for values of voltage above 6.5V This would mean that more period doublings were present, so the relationship between voltage and force could be analysed more fully. Investigating the pendulum when it is driven at forces that are inside the 'chaotic realm' would also be interesting.

It was unfortunate that the apparatus worked itself loose near the end of the available lab time, such that there was not enough time to fix it, because it would have been useful to have collected more results for the second experiment.

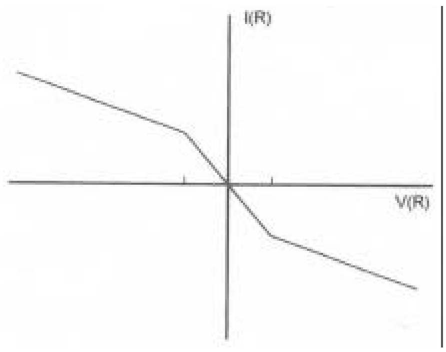
Further Experiments

Electronic implementation of a non-linear system

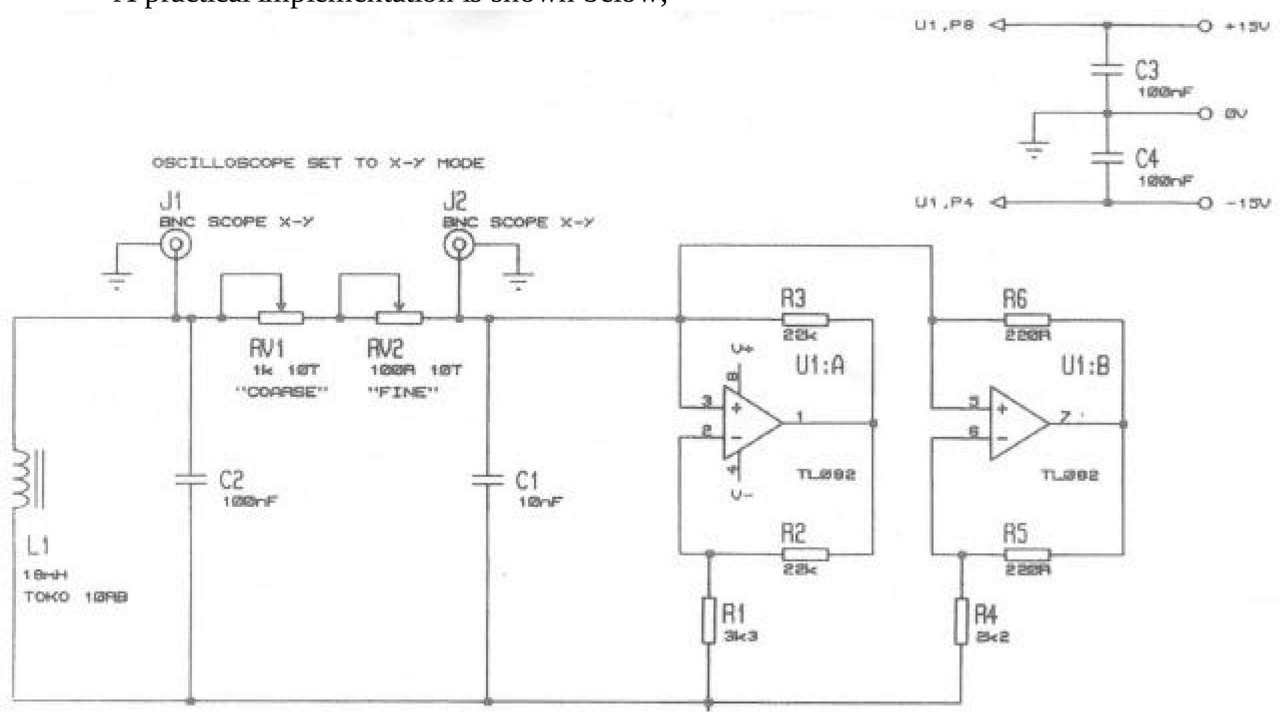
An electronic implementation of a non-linear system could also be set up as mentioned in the diary. The schematics relating to this are shown below;



The resistor in this circuit must be non-linear. The ideal characteristics for the resistor are shown in the following graph;



A practical implementation is shown below;



The oscilloscope will show attractors, bifurcations and period doublings of the system.

Investigating other types of pendulum

The executive toy that was mentioned in the diary could be examined for signs of chaotic behavior, especially its sensitivity to initial conditions, by counting the number of revolutions of the circles that are driven by the pendulum for different starting angles.

Summing up

The investigation was successful in satisfying the statements in the aim that was set out at the beginning of the investigation. Although there is plenty more that can be investigated and many more results that could have been collected, there was insufficient time available to extend the investigation any further.

Bibliography

James Gleick

CHAOS - Making a new science

ISBN 0-434-29554-X

Karl - Heinz Becker and Michael Dörfler

Dynamical systems and fractals

Cambridge University Press

First Published 1989

Reprinted 1990 (three times), 1991

ISBN 0-521-36910-X

Various websites