

Aim

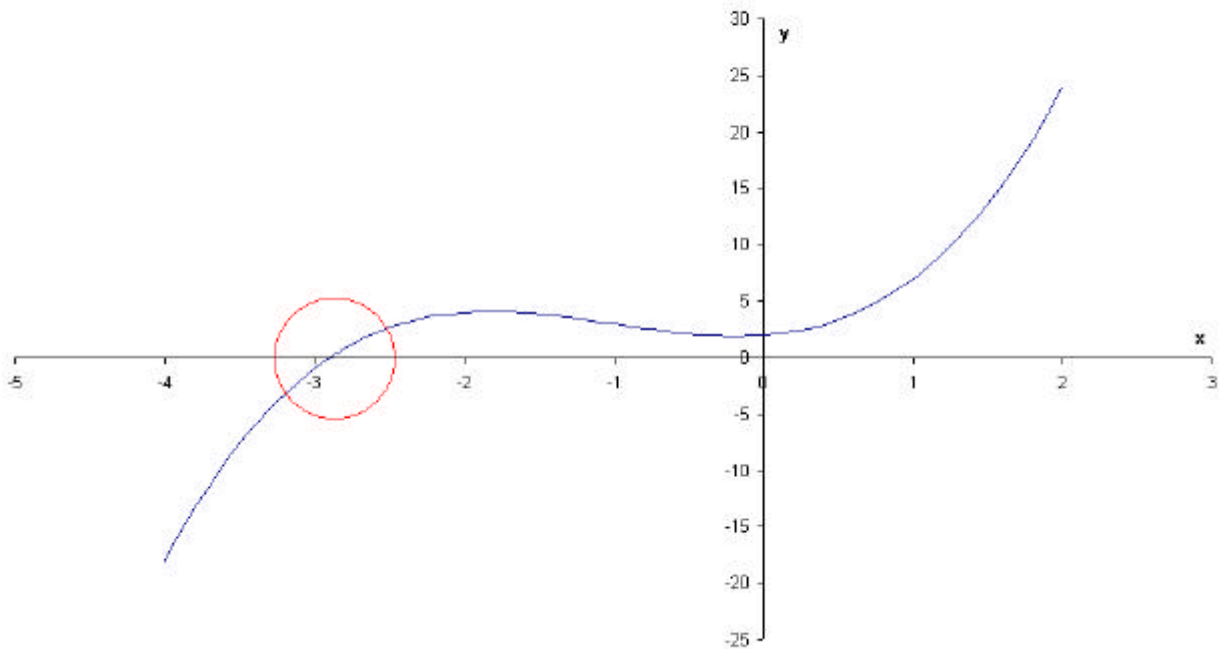
The aim of this investigation is to use and compare three different methods of finding specific solutions to polynomials that cannot be solved algebraically. The methods that will be considered are; decimal search, Newton - Raphson and rearrangement of the equation. The advantages, disadvantages and limitations of each method will be discussed.

Decimal Search

The decimal search works by evaluating the function in 0.1 step increments between two limits. When a change of sign occurs, the function is evaluated in 0.01 step increments between the two points where the sign change occurred. This continues until the required level of accuracy is reached. The interval in which the search is initially started can be determined from a graph of the function.

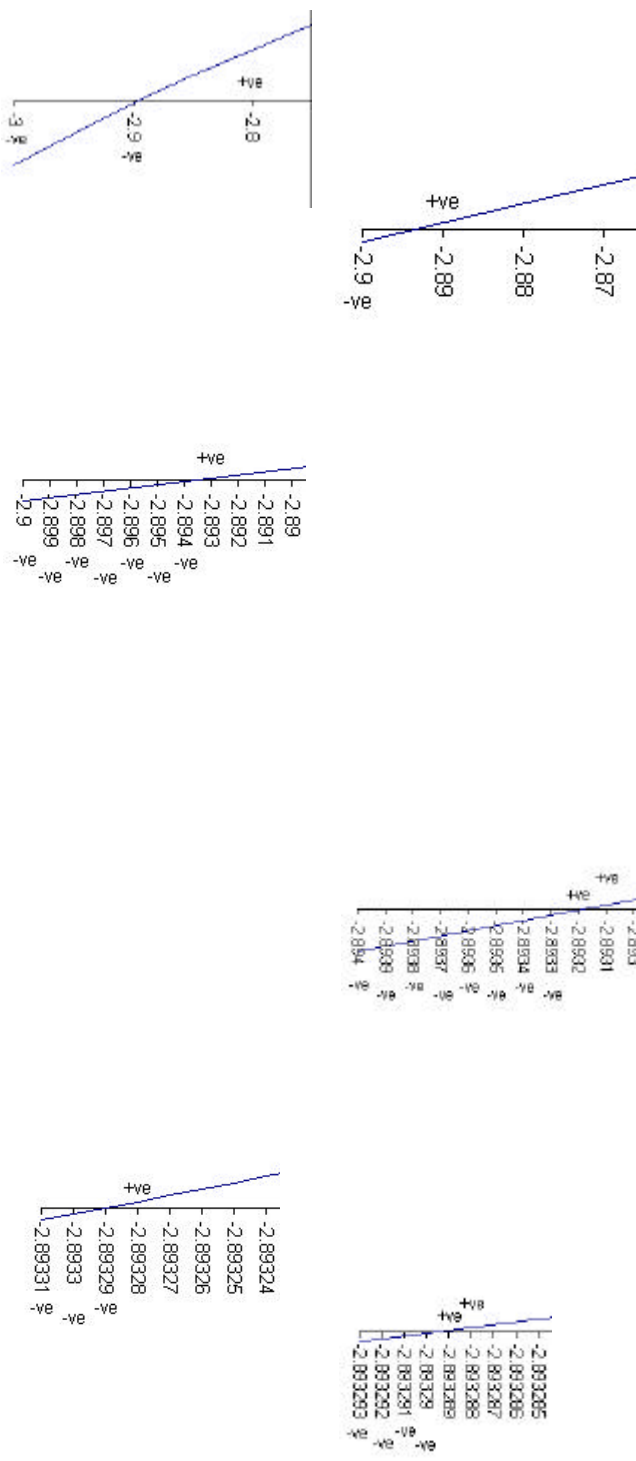
A decimal search succeeds in finding the solution between -2 and -3 for the following function; $f(x) = x^3 + 3x^2 + x + 2$

$$y = x^3 + 3x^2 + x + 2$$



An Excel 'macro' was used to do the calculations. A copy of the script can be found in Appendix A.

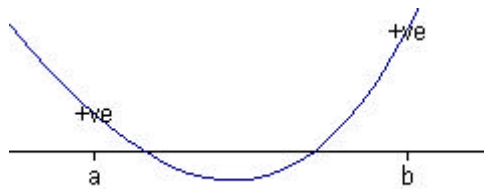
x	$f(x)$
-3.0	-1.000000
-2.9	-0.059000
-2.8	0.768000
-2.90	-0.059000
-2.89	0.028731
-2.900	-0.059000
-2.899	-0.050176
-2.898	-0.041363
-2.897	-0.032561
-2.896	-0.023771
-2.895	-0.014992
-2.894	-0.006225
-2.893	0.002531
-2.8940	-0.006225
-2.8939	-0.005349
-2.8938	-0.004473
-2.8937	-0.003597
-2.8936	-0.002721
-2.8935	-0.001846
-2.8934	-0.000970
-2.8933	-0.000095
-2.8932	0.000781
-2.89330	-0.000095
-2.89329	-0.000007
-2.89328	0.000081
-2.89329	-0.000007
0	
-2.89328	0.000002
9	



From these calculations, it can be seen that for $f(x)=0$

$$x = -2.893285 \pm 0.0000005$$

The decimal search works for most types of polynomial. However, if the polynomial has more than one solution between two integers, then at least one of the solutions cannot be found by using the decimal search in this manner. If there are an even number of solutions between two integers and a graph of the function is not plotted, then no solutions will be found. This is because

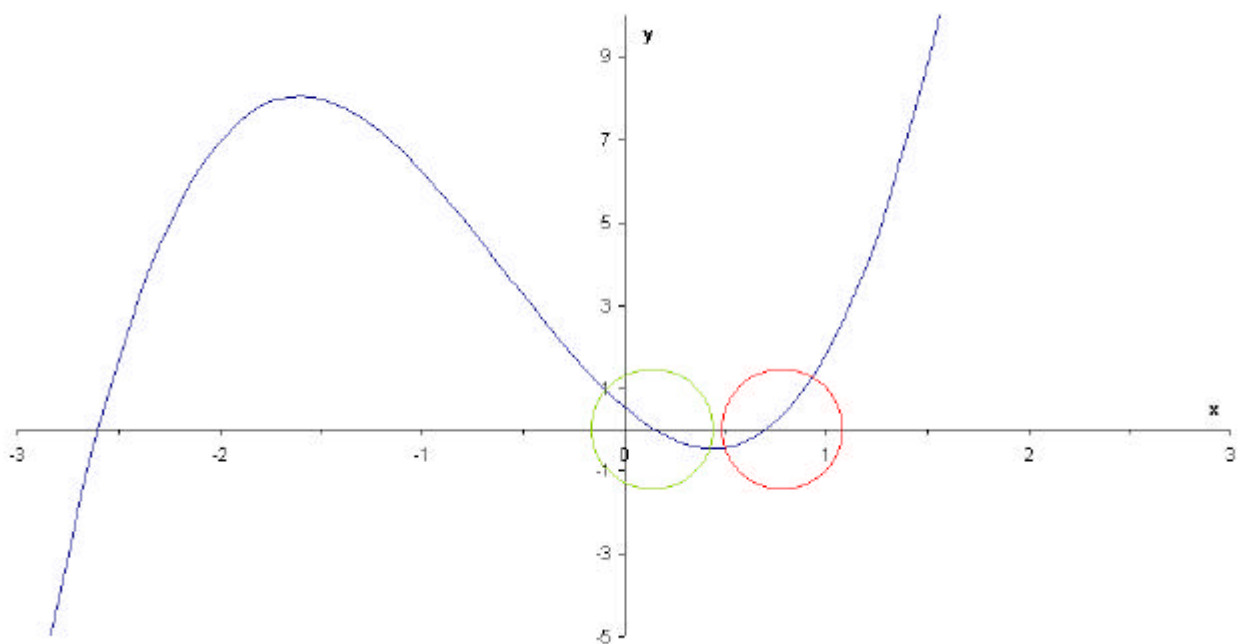


when the interval search is conducted - to find the interval in which the decimal search will operate - there will be no change of sign, so both solutions will be missed.

If however, a graph is plotted, only the solution on the left can be found because the decimal search will stop after it has found one solution. If, instead, the decimal search is run from right to left, then only the solution on the right can be found. When there are more than two solutions in an interval, the solution in the middle cannot be found by using a decimal search

A decimal search fails in finding the solution between 0.5 and 1 for the following function;
 $f(x) = 2x^3 + 3.5x^2 - 4.21x + 0.546$

$$y = 2x^3 + 3.5x^2 - 4.21x + 0.546$$



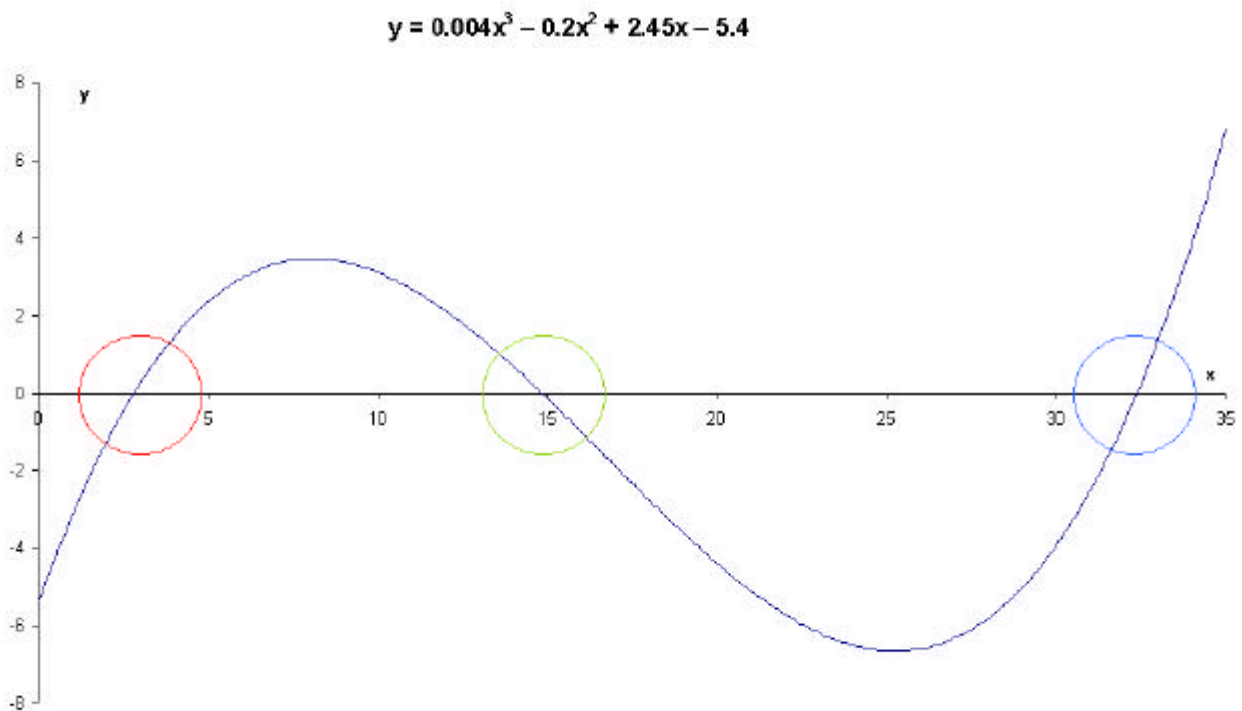
x	f(x)
0	0.546000
0.1	0.162000
0.2	-0.140000
0.1	0.162000
0.11	0.127912
0.12	0.094700
0.13	0.062244
0.14	0.030688
0.15	0.000000
0.16	-0.029800

For this function, although an exact solution has been found, it is not the solution that was required. Therefore, the decimal search has failed.

Newton - Raphson Method

The Newton - Raphson method uses an iterative formula to find the roots of an equation. This means that an estimation of the x value of root is made. The equation (map) is then evaluated with this value. The evaluation yields another value of x which is then used to evaluate the map again. The map is iterated in this way until it yields an answer to an appropriate level of accuracy or it is clearly diverging from the root. It will soon become clear if the method is failing because if it is succeeding, then the iterations usually converge on the answer very rapidly - i.e. after only a few iterations. To find the first value of x, a graph should be drawn. A value of x that is near to the root can then be chosen.

The Newton - Raphson method will now be used to find all of the solutions to the following equation; (The root will be taken when $x_n = x_{n+1} = x_{n+2}$)



The iterative formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Therefore, for this equation, $f(x) = 0.004x^3 - 0.2x^2 + 2.45x - 5.4$

$$f'(x) = 0.012x^2 - 0.4x + 2.45$$

An Excel spreadsheet was used to do the calculations. A copy of the spreadsheet configuration can be found in Appendix A.

The first root that will be found is the one between $x=0$ and $x=5$

$$x_1 = 2$$

n	x_n	x_{n+1}
1	2	2.746761
2	2.746761	2.813676
3	2.813676	2.814203
4	2.814203	2.814203
5	2.814203	2.814203
6	2.814203	2.814203

From these calculations, it can be seen that for $f(x)=0$

$$x = 2.814203 \pm 0.0000005$$

The second root that will be found is the one between $x=10$ and $x=15$

$$x_1 = 15$$

n	x_n	x_{n+1}
1	15	14.82353
2	14.82353	14.82276
3	14.82276	14.82276
4	14.82276	14.82276
5	14.82276	14.82276
6	14.82276	14.82276

From these calculations, it can be seen that for $f(x)=0$

$$x = 14.82276 \pm 0.0000005$$

The last root that will be found is the one between $x=30$ and $x=35$

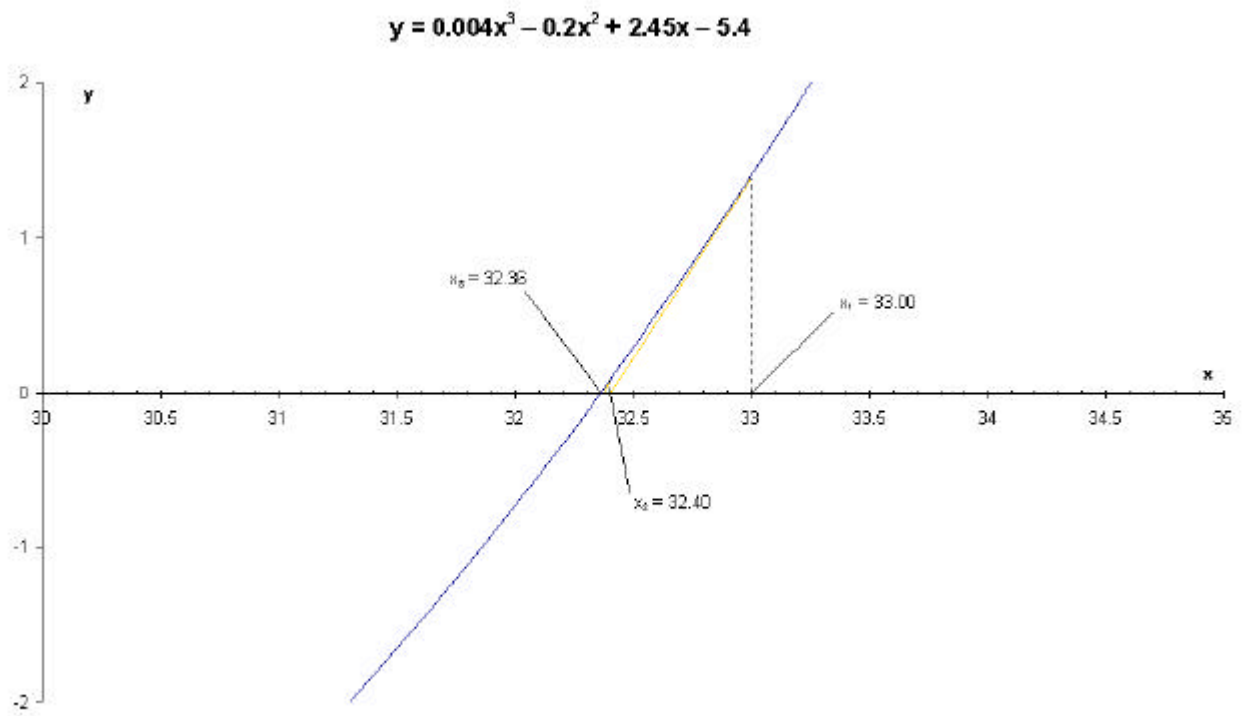
$$x_1 = 33$$

n	x_n	x_{n+1}
1	33	32.39689
2	32.39689	32.36314
3	32.36314	32.36303
4	32.36303	32.36303
5	32.36303	32.36303
6	32.36303	32.36303

From these calculations, it can be seen that for $f(x)=0$

$$x = 32.36303 \pm 0.0000005$$

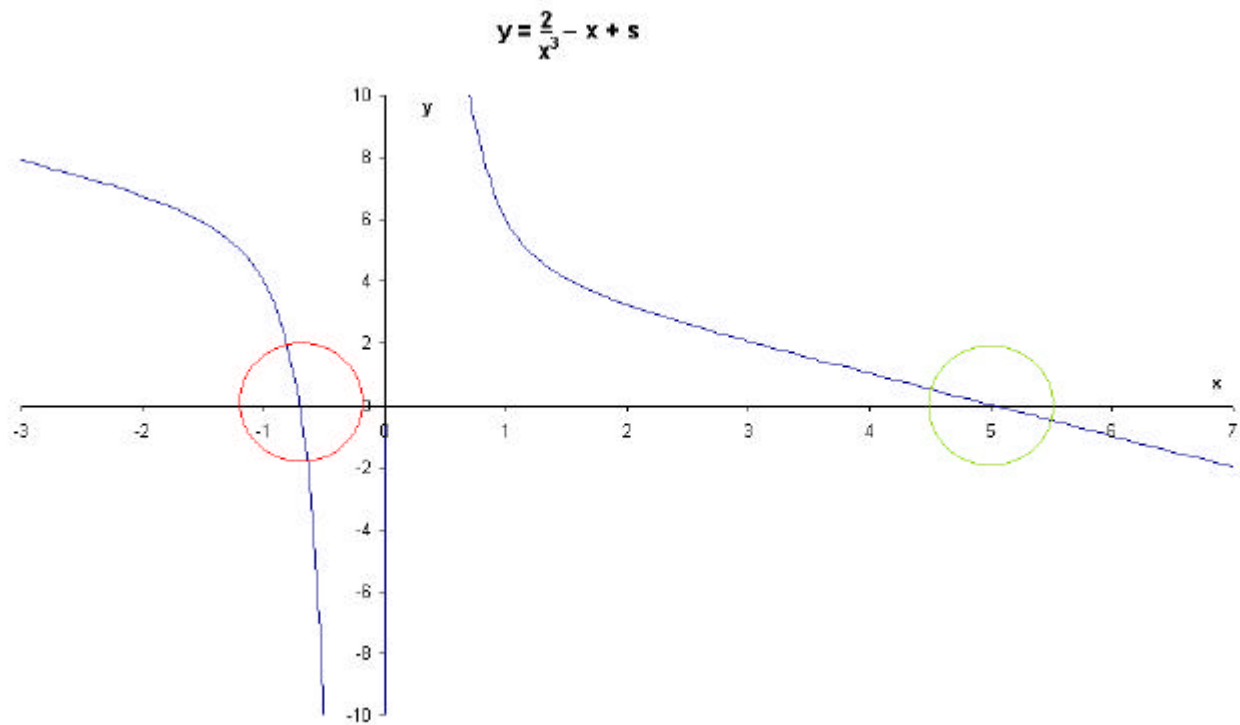
Graphical Illustration Of The Newton - Raphson Method.



The Newton - Raphson Method of finding the root of an equation will fail when the initial value is too far away from the root. This method will also fail if there is a turning point near to the root. This will cause the formula to converge to another root of the equation. The Newton - Raphson method will also fail if the root that is required is near to a discontinuity in the equation.

The Newton - Raphson method will fail for the following equation;

$$f(x) = \frac{2}{x^3} - x + 5$$



The root that will be found is the one between $x = -2$ and $x = 0$

$$x_1 = -1.5$$

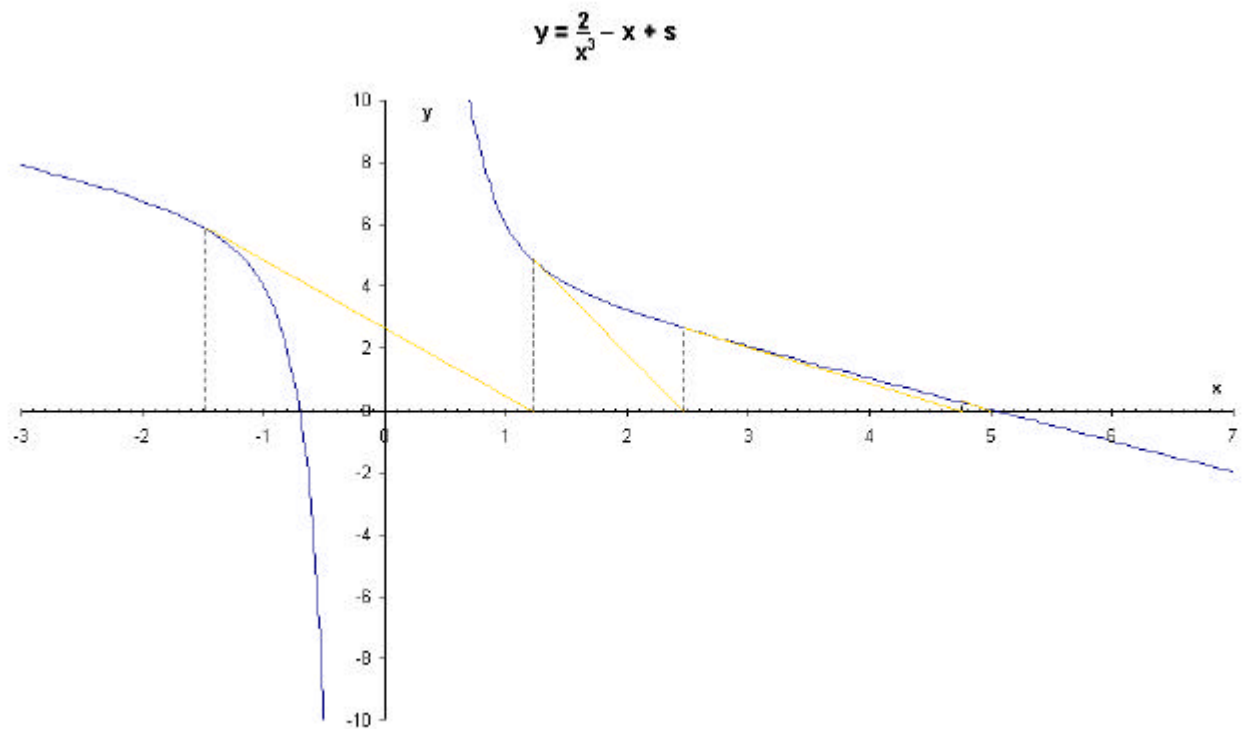
n	x_n	x_{n+1}
1	-1.5	1.2
2	1.2	2.483938
3	2.483938	4.770163
4	4.770163	5.015582
5	5.015582	5.015849
6	5.015849	5.015849

From these calculations, it can be seen that for $f(x) = 0$

$$x = 5.015849 \pm 0.0000005$$

This is not the solution that was being searched for and therefore, the calculation has failed.

Explanation Of Failure



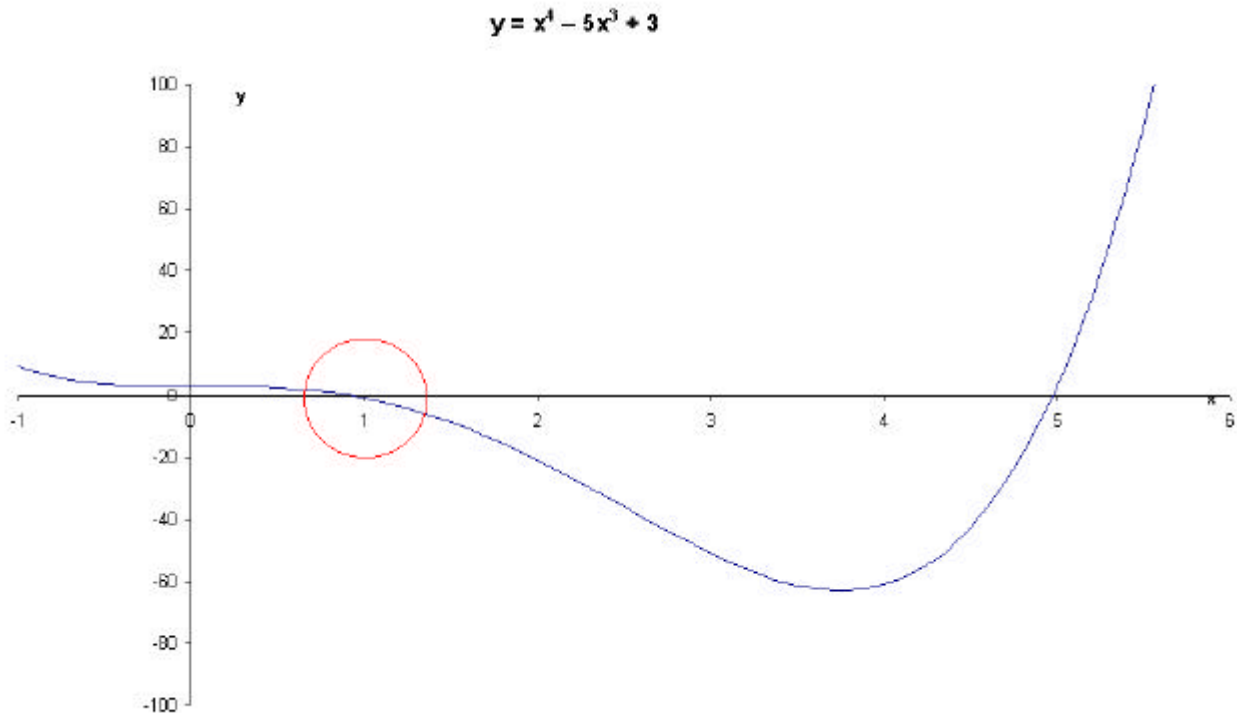
The calculation failed because the gradient of the equation when $x_1 = -1.5$ is small enough to allow the tangent to cross the discontinuity. The curve slopes in the opposite direction on the positive side of the discontinuity. This slope means that the tangent cannot cross back onto the negative side of the discontinuity. This means that the tangent will converge to the wrong root. If a value for x had been chosen where the gradient was steep enough to prevent the tangent from crossing the discontinuity before it crossed the x -axis, then the correct solution would have been found.

Rearranging The Equation To Find A Root

This method of rearranging the equation works in a similar way to the Newton - Raphson method. A function, $f(x)$ is rearranged into the form of $x=g(x)$. This new function is then iterated in the following way; $x_{n+1}=g(x_n)$.

One solution to the following equation is shown below;

$$f(x) = x^4 - 5x^3 + 3$$



A possible rearrangement is $g(x) = \sqrt[3]{\frac{x^4 + 3}{5}}$

Therefore, for this equation, the iterative formula is $x_{n+1} = \sqrt[3]{\frac{x_n^4 + 3}{5}}$

An Excel spreadsheet was used to do the calculations. A copy of the spreadsheet configuration can be found in Appendix A.

The root that will be found is the one between $x=0$ and $x=1$

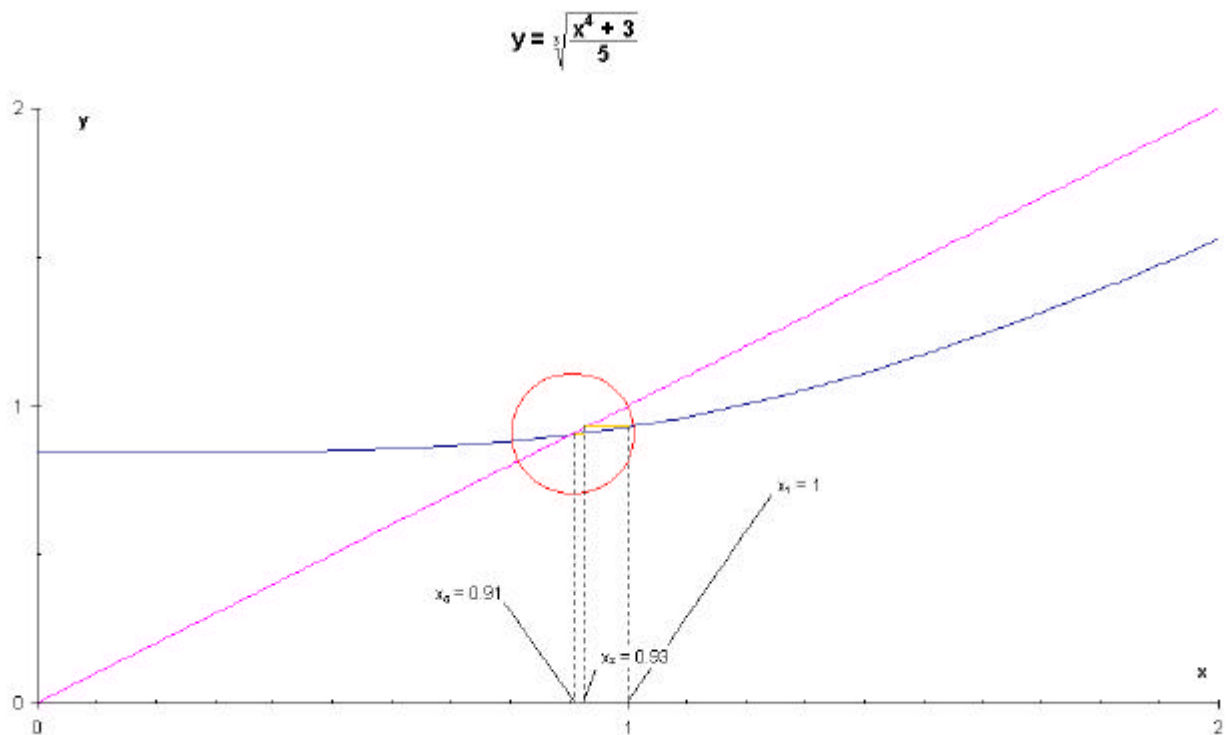
$$x_1 = 1$$

n	x_n	x_{n+1}
1	1	0.928318
2	0.928318	0.907967
3	0.907967	0.902842
4	0.902842	0.901596
5	0.901596	0.901296
6	0.901296	0.901224
7	0.901224	0.901206
8	0.901206	0.901202
9	0.901202	0.901201
10	0.901201	0.901201
11	0.901201	0.901201
12	0.901201	0.901201

From these calculations, it can be seen that for $f(x)=0$

$$x=0.901201$$

Graphical Illustration Of The Rearrangement Method



If the first derivative of the function $g(x) = \sqrt[3]{\frac{x^4 + 3}{5}}$ is evaluated for values that are near to the solution, it can be seen that the gradient of the equation at these points is between -1 and $+1$.

If the gradient was outside of these limits, then the process of iterating this function would not cause the values produced to converge onto the root. - They would diverge away from the root.

The following example demonstrates this;

The same function and rearrangement will be used, but an attempt will be made to discover the other root of the equation.

$$f(x) = x^4 - 5x^3 - 3$$

$$g(x) = \sqrt[3]{\frac{x^4 - 3}{5}}$$

$$x_{n+1} = \sqrt[3]{\frac{x_n^4 - 3}{5}}$$

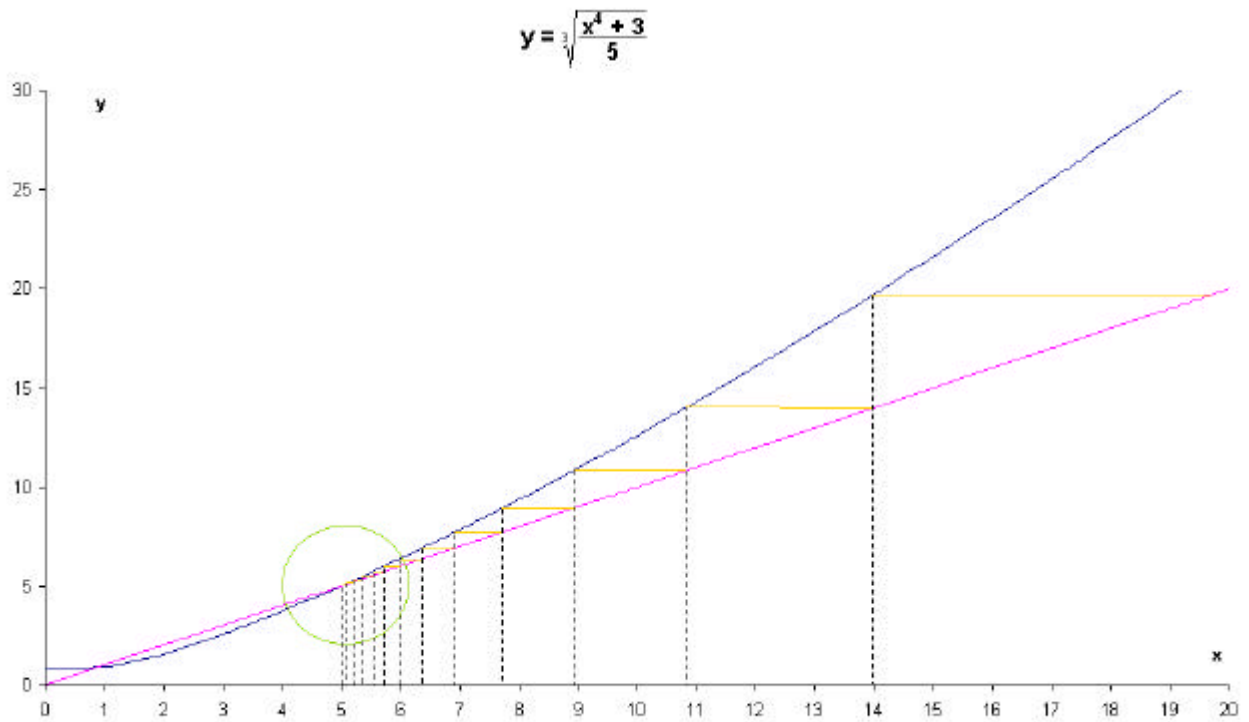
The root that will be found is the one between $x=4$ and $x=5$

$$x_1 = 4.5$$

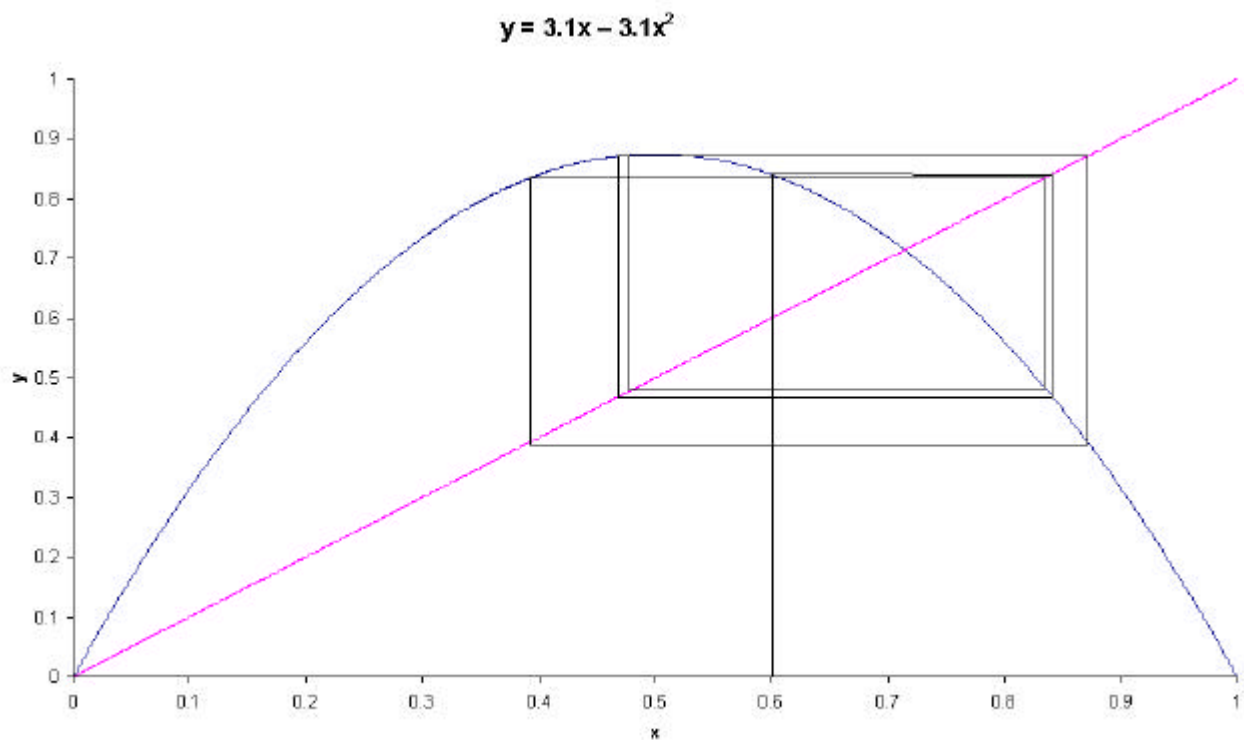
<i>n</i>	<i>x_n</i>	<i>x_{n+1}</i>
1	5	5.007987
2	5.007987	5.018606
3	5.018606	5.032732
4	5.032732	5.05154
5	5.05154	5.07661
6	5.07661	5.110077
7	5.110077	5.154843
8	5.154843	5.21488
9	5.21488	5.295681
10	5.295681	5.404932
11	5.404932	5.553563
12	5.553563	5.757422
13	5.757422	6.040072
14	6.040072	6.437621
15	6.437621	7.007502
16	7.007502	7.845265
17	7.845265	9.118738
18	9.118738	11.14258
19	11.14258	14.55523
20	14.55523	20.78318
21	20.78318	33.41655
22	33.41655	62.9447
23	62.9447	146.4274
24	146.4274	451.3426
25	451.3426	2024.658
26	2024.658	14978.87
27	14978.87	215931.1
28	215931.1	7575832

After 28 iterations, it is clear that the a value for the root of the equation is not appearing

and the values are in fact diverging from the root. A graphical illustration of what is happening is shown below;



There are two effects that an iterative formula can have. The iterations can produce a staircase diagram as shown above, or they can produce a cobweb diagram as shown below;



At first, each iteration yields a unique value of x , but after a few iterations it can be seen that that previous values of x are returned to. If the family of functions where $y = ax - ax^2$ is iterated for many values of x and then a graph of a against the values of x that are constantly repeated when the map is iterated is plotted a pattern is produced. This pattern can be described as chaotic. Chaotic properties include apparent randomness. If however the pattern is investigated, it can be seen that the overall pattern includes itself many times. The map of $y = ax - ax^2$ is called the *Logistic Map*. Another example of a chaotic pattern is the *Mandelbrot set*. Many functions can produce chaotic patterns The pattern produced by the *Logistic Map* is shown on the next page. The repeating pattern can be easily seen.

The cause of the dispersion in the function $g(x) = \sqrt{\frac{x^4 - 3}{5}}$ is due to the fact that the gradient of the equation is greater than 1 at points near to the intersection of the line $y = x$.

Comparison Of Methods

In order to compare the methods that have been investigated, the root between $x=0$ and $x=1$ of the function $f(x) = x^4 - 5x^3 + 3$ shall be found using the Decimal Search Method and the Newton - Raphson Method.

Decimal Search

The decimal search will search in the interval $0 \leq x \leq 1$

x	$f(x)$
0	3.0000000
0.1	2.9951000
0.2	2.9616000
0.3	2.8731000
0.4	2.7056000
0.5	2.4375000
0.6	2.0496000
0.7	1.5251000
0.8	0.8496000
0.9	0.0111000
1	-1.0000000
0.9	0.0111000
0.91	-0.0821054
0.9	0.0111000
0.901	0.0018574
0.902	-0.0074026
0.901	0.0018574
0.9011	0.0009321
0.9012	0.0000068
0.9013	-0.0009188
0.9012	0.0000068
0.90121	-0.0000858
0.9012	0.0000068
0.901201	-0.0000025

From these calculations, it can be seen that for $f(x) = 0$

$$x = 0.901201 \pm 0.0000005$$

Due to the fact that the solution to the equation is near to the end of the interval, a large number of steps were needed to find the solution to the required level of accuracy.

Newton - Raphson Method

$$x_1 = 1$$

n	x_n	x_{n+1}
1	1	0.909091
2	0.909091	0.901258
3	0.901258	0.901201
4	0.901201	0.901201
5	0.901201	0.901201
6	0.901201	0.901201

From these calculations, it can be seen that for $f(x)=0$

$$x=0.901201 \pm 0.0000005$$

This method only took 3 steps to find the solution to the equation, however, a further 3 steps had to be performed to ensure that the iterative formula had settled.

Comparison Of Methods

Rearranging The Equation

Number of steps required: 12

Decimal Search

Number of steps required: 24

Newton - Raphson

Number of steps required: 6

Conclusion

Each of the methods was easily conducted with the aid of a computer. Both the Newton - Raphson method and Rearranging The Equation can easily be conducted on a scientific calculator. It would however, be quite difficult to conduct the Decimal Search on a scientific calculator. This is because the values of the x in the equation is not related to the value obtained when the function is evaluated. - The decimal search is however the "best" method to use to find a root of an equation when a computer is being used. This is because only the function needs to be provided to the program. - First derivatives, rearrangements etc. do not have to be calculated. Also, if the method fails the user is told that no solutions have been found, whereas with the other methods the value for the root diverges away from the real root and the value is presented as a successful answer. i.e. If the decimal search fails then it simply does not work. If any of the other methods fail they still appear to produce an answer. A computer would be unable to decide whether the method had failed or had been successful without further analysing the results. For manual calculation however, the Newton - Raphson method appears to be the most attractive. This method, and the rearrangement method, however, both require the original function to be manipulated before any progress can be made. It may also be hard to deduce whether the method will succeed or fail just by inspecting the graph. With the decimal search method the start and end points of the search can be decided by looking at the graph with respect to the other roots of the equation, and possible difficulties can be immediately identified.

Appendix A

Decimal Search

This is the script used to perform the decimal search;

```
Sub decimal_search()
```

```
  a = 0           'Start of search interval  
  b = 1           'End of search interval  
  s = 0.1         'Initial step  
  y = 6           'Number of decimal places
```

```
start:
```

```
For x = a To b Step s
```

```
  f = (x ^ 4) - (5 * x ^ 3) + 3           'The function that is being solved
```

```
  'Put some information in the spreadsheet
```

```
  ActiveCell.FormulaR1C1 = x  
  ActiveCell.Offset(0, 1).Range("A1").Select  
  ActiveCell.FormulaR1C1 = f  
  ActiveCell.Offset(1, -1).Range("A1").Select
```

```
  If x = a Then                               'Find out if this is the first evaluation  
    If f = Abs(f) Then t = 0 Else t = 1       'If so, discover the sign of the function  
  End If
```

```
  If t = 0 Then                               'If the function was positive on the first iteration  
    If f <> Abs(f) Then GoTo nexter           'Then stop if it is now negative  
  End If
```

```
  If t = 1 Then                               'If the function was negative on the first iteration  
    If f = Abs(f) Then GoTo nexter           'Then stop if it is now positive  
  End If
```

```
Next x                                         'Increment x and evaluate the function again
```

```
MsgBox "No roots found in interval"           'The whole interval had been checked and there  
End                                           'was no change of sign, so tell user and end
```

```
nexter:                                       'There must have been a change of sign
```

```
ActiveCell.Offset(1, 0).Range("A1").Select   'Move around the spreadsheet  
b = x                                         'Set upper limit to current value of x  
a = x - s                                     'Set lower limit to previous value of x  
s = s / 10                                    'Increase the level of accuracy by one decimal place  
z = z + 1                                     'Record the number of decimal places that have been reached  
If z = y Then End                             'If the required number of dp's have been reached then stop  
GoTo start                                    'If not, then keep going
```

```
End Sub
```


Newton - Raphson

This is the spreadsheet configuration used to perform the Newton - Raphson calculation.

The example given is for $f(x) = x^4 - 5x^3 - 3$

x_n	x_{n+1}
1	=A4 - (((A4^4) - (5*A4^3) + 3) / ((4*A4^3)-(15*A4^2)))
=B4	=A5 - (((A5^4) - (5*A5^3) + 3) / ((4*A5^3)-(15*A5^2)))
=B5	=A6 - (((A6^4) - (5*A6^3) + 3) / ((4*A6^3)-(15*A6^2)))
=B6	=A7 - (((A7^4) - (5*A7^3) + 3) / ((4*A7^3)-(15*A7^2)))
=B7	=A8 - (((A8^4) - (5*A8^3) + 3) / ((4*A8^3)-(15*A8^2)))
=B8	=A9 - (((A9^4) - (5*A9^3) + 3) / ((4*A9^3)-(15*A9^2)))
=B9	=A10 - (((A10^4) - (5*A10^3) + 3) / ((4*A10^3)-(15*A10^2)))
=B10	=A11 - (((A11^4) - (5*A11^3) + 3) / ((4*A11^3)-(15*A11^2)))
=B11	=A12 - (((A12^4) - (5*A12^3) + 3) / ((4*A12^3)-(15*A12^2)))
=B12	=A13 - (((A13^4) - (5*A13^3) + 3) / ((4*A13^3)-(15*A13^2)))
=B13	=A14 - (((A14^4) - (5*A14^3) + 3) / ((4*A14^3)-(15*A14^2)))
=B14	=A15 - (((A15^4) - (5*A15^3) + 3) / ((4*A15^3)-(15*A15^2)))
=B15	=A16 - (((A16^4) - (5*A16^3) + 3) / ((4*A16^3)-(15*A16^2)))
=B16	=A17 - (((A17^4) - (5*A17^3) + 3) / ((4*A17^3)-(15*A17^2)))
=B17	=A18 - (((A18^4) - (5*A18^3) + 3) / ((4*A18^3)-(15*A18^2)))
=B18	=A19 - (((A19^4) - (5*A19^3) + 3) / ((4*A19^3)-(15*A19^2)))
=B19	=A20 - (((A20^4) - (5*A20^3) + 3) / ((4*A20^3)-(15*A20^2)))
=B20	=A21 - (((A21^4) - (5*A21^3) + 3) / ((4*A21^3)-(15*A21^2)))
=B21	=A22 - (((A22^4) - (5*A22^3) + 3) / ((4*A22^3)-(15*A22^2)))
=B22	=A23 - (((A23^4) - (5*A23^3) + 3) / ((4*A23^3)-(15*A23^2)))
=B23	=A24 - (((A24^4) - (5*A24^3) + 3) / ((4*A24^3)-(15*A24^2)))
=B24	=A25 - (((A25^4) - (5*A25^3) + 3) / ((4*A25^3)-(15*A25^2)))
=B25	=A26 - (((A26^4) - (5*A26^3) + 3) / ((4*A26^3)-(15*A26^2)))

Rearranging The Equation

This is the spreadsheet configuration used to perform the calculation for the rearrangement method.

The example given is for $f(x) = x^4 - 5x^3 - 3$ $g(x) = \sqrt[3]{\frac{x^4 - 3}{5}}$

x_n	x_{n+1}
5	$(((A2^4)+3)/5)^{1/3}$
=B2	$(((A3^4)+3)/5)^{1/3}$
=B3	$(((A4^4)+3)/5)^{1/3}$
=B4	$(((A5^4)+3)/5)^{1/3}$
=B5	$(((A6^4)+3)/5)^{1/3}$
=B6	$(((A7^4)+3)/5)^{1/3}$
=B7	$(((A8^4)+3)/5)^{1/3}$
=B8	$(((A9^4)+3)/5)^{1/3}$
=B9	$(((A10^4)+3)/5)^{1/3}$
=B10	$(((A11^4)+3)/5)^{1/3}$
=B11	$(((A12^4)+3)/5)^{1/3}$
=B12	$(((A13^4)+3)/5)^{1/3}$
=B13	$(((A14^4)+3)/5)^{1/3}$
=B14	$(((A15^4)+3)/5)^{1/3}$
=B15	$(((A16^4)+3)/5)^{1/3}$
=B16	$(((A17^4)+3)/5)^{1/3}$
=B17	$(((A18^4)+3)/5)^{1/3}$
=B18	$(((A19^4)+3)/5)^{1/3}$
=B19	$(((A20^4)+3)/5)^{1/3}$
=B20	$(((A21^4)+3)/5)^{1/3}$
=B21	$(((A22^4)+3)/5)^{1/3}$
=B22	$(((A23^4)+3)/5)^{1/3}$
=B23	$(((A24^4)+3)/5)^{1/3}$